Journal of Engineering Physics and Thermophysics, Vol. 73, No. 2, 2000

FREE CONVECTION WITH A NONLINEAR DEPENDENCE OF DENSITY ON TEMPERATURE: PLANE PROBLEMS

V. N. Korovkin and A. P. Andrievskii

UDC 536.25

Results of numerical simulation of free-convective heat transfer with a square-law dependence of density on temperature are presented. The special features of the velocity and temperature fields above a horizontal linear heat source and on a flat vertical surface as a function of the boundary conditions and the Prandtl number are studied. Detailed tables of numerical solutions are given.

Free convection has been the object of comprehensive investigation for a long time, being of both practical and scientific interest [1]. In the majority of theoretical studies, a linear dependence of density on temperature is used. However, for many fluids, e.g., water, molten bismuth, gallium, tellurium, the dependence of ρ on T has an extremum [2].

In what follows, we present results of a complex numerical investigation of fully developed free-convective flows above a horizontal linear heat source and on a flat impermeable vertical semiinfinite plate for three types of thermal boundary conditions: an adiabatic surface, a constant temperature, and a constant heat flux on the surface.

An analysis is made within the framework of the model of a stationary laminar boundary layer in the Boussinesq approximation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,,\quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\gamma\,(T-T_\infty)^2\,,\quad u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{\Pr}\frac{\partial^2 T}{\partial y^2} \tag{1}$$

with the corresponding boundary conditions for the velocity and temperature fields.

Isothermal Surface. Let a heated vertical plate with the temperature T_w be placed in an infinite medium with constant properties and the temperature T_∞. We direct the x axis upward along the plate and the y axis along the normal to it. Then the boundary conditions are expressed as

$$\begin{aligned} &y=0:\quad v=u=0\,,\quad T=T_w\,,\\ &y\to\infty:\quad u\to 0\,,\quad T\to T_\infty\,. \end{aligned} \tag{2}$$

We transform the system of equations (1)-(2) to a new form by passing from the coordinates x and y to the variables x and η, where

$$\eta=\left(\frac{g\gamma\Delta T_w^2}{\nu^2}\right)^{1/4} x^{-1/4}\,y\,. \tag{3}$$

Moreover, a reduced stream function and a dimensionless temperature are introduced:

Polotsk State University, Novopolotsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 2, pp. 381-386, March–April, 2000. Original article submitted January 26, 1999.

$$\Psi = (g\gamma\nu^2\Delta T_w^2)^{1/4} f(\eta)\, x^{3/4}, \quad \Delta T = \Delta T_w h(\eta). \tag{4}$$

The resultant system is written in the form

$$f''' + \frac{3}{4} f f'' - \frac{1}{2} f'^2 + h^2 = 0, \quad \frac{1}{\Pr} h'' + \frac{3}{4} f h' = 0,$$

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 0, \quad h(0) = 1, \quad h(\infty) = 0. \tag{5}$$

Adiabatic Surface. In considering free-convective heat transfer on an adiabatic vertical flat plate with a linear heat source embedded in the leading edge, we have the following boundary conditions:

$$y = 0: \quad v = u = 0, \quad \frac{\partial T}{\partial y} = 0,$$

$$y \to \infty: \quad u \to 0, \quad T \to T_\infty. \tag{6}$$

Moreover, the law of energy conservation requires that at any $x > 0$ the energy transferred by convection be equal to the energy

$$Q_0 = \rho C_p \int_0^\infty u \Delta T\, dy = \text{const}, \tag{7}$$

liberated by the linear source.

Introducing transformations of the form

$$\Psi = \left(\frac{g\gamma\nu^2 Q_0^2}{\rho^2 C_p^2}\right)^{1/6} f(\eta)\, x^{1/2}, \quad \eta = \left(\frac{g\gamma Q_0^2}{\rho^2 C_p^2 \nu^4}\right)^{1/6} x^{-1/2} y,$$

$$\Delta T = \left(\frac{Q_0^4}{\rho^4 C_p^4 g\gamma\nu^2}\right)^{1/6} h(\eta)\, x^{-1/2}, \tag{8}$$

we arrive at the following boundary-value problem:

$$f''' + \frac{1}{2} f f'' + h^2 = 0, \quad \frac{1}{\Pr} h'' + \frac{1}{2} f h' + \frac{1}{2} f' h = 0, \quad \int_0^\infty f' h\, d\eta = 1,$$

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 0, \quad h'(0) = 0, \quad h(\infty) = 0. \tag{9}$$

Constant Heat Flux on the Surface. In this case, a solution of Eqs. (1) with the boundary conditions

$$y = 0: \quad v = u = 0, \quad -k \frac{\partial T}{\partial y} = q_w = \text{const},$$

$$y \to \infty: \quad u \to 0, \quad T \to T_\infty$$

is sought in the form

$$\Psi = \left(\frac{g\gamma\nu^4 q_w^2}{k^2} \right)^{1/6} f(\eta)\, x^{5/6}, \quad \eta = \left(\frac{g\gamma q_w^2}{k^2 \nu^2} \right)^{1/6} x^{-1/6} y,$$

$$\Delta T = \frac{q_w}{k} \left(\frac{k^2 \nu^2}{g\gamma q_w^2} \right)^{1/6} h(\eta)\, x^{1/6}, \tag{10}$$

where the functions f and h satisfy the following system of equations:

$$f''' + \frac{5}{6} ff'' - \frac{2}{3} f'^2 + h^2 = 0, \quad \frac{1}{\Pr} h'' + \frac{5}{6} fh' - \frac{1}{6} f' h = 0,$$

$$f(0) = 0, \; f'(0) = 0, \; f'(\infty) = 0, \; h'(0) = -1, \; h(\infty) = 0. \tag{11}$$

Free Convection above a Linear Heat Source. The boundary conditions for this problem have the form

$$y = 0: \; v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0,$$

$$y \to \infty: \; u \to 0, \; T \to T_\infty. \tag{12}$$

Substituting expressions (8), where

$$Q_0 = \rho C_p \int_{-\infty}^{+\infty} u \Delta T \, dy = \text{const}, \tag{13}$$

into (1) and taking into account (12), we obtain

$$f''' + \frac{1}{2} ff'' + h^2 = 0, \quad \frac{1}{\Pr} h'' + \frac{1}{2} fh' + \frac{1}{2} f' h = 0,$$

$$f(0) = f''(0) = h'(0) = 0, \; f'(\infty) = 0, \; h(\infty) = 0, \; \int_0^\infty f' h \, d\eta = 1/2. \tag{14}$$

Of all the problems considered above, only (14) allows an analytical solution in quadratures for two values of the Pr number equal to 1 and 1/3 [3]. The latter allows a numerical analysis of (14) "with one's eyes open" using analytical relations [3] as a proving ground for testing the accuracy of the constructed numerical solutions. This is of prime importance, since, in spite of the simplicity of the ordinary differential equations that describe the behavior of various convective fluid flows within the framework of the classical assumptions, construction of numerical solutions of these equations is not a simple matter. It is difficult to find the latter because the behavior of the functions f, f', and h for $\eta \to \infty$ differs greatly, and the necessity of introducing three additional parameters aggravates the difficulties of the analysis. That is probably why the numerical data obtained by different authors [4-7] in integrating system (5) are scattered.

Calculation Results. Four nonlinear two-point boundary-value problems were solved numerically by reducing (5), (9), (11), and (14) to the corresponding Cauchy problems, which are stated at $\eta = 0$. The Cauchy problem, which requires the introduction of two additional parameters $f''(0)$ (or $f'(0)$) and $h'(0)$ (or $h(0)$), can be related to the initial boundary-value problem by integrating the system of ordinary differential equations for finding the unknown functions f and h in the η-direction by the Runge–Kutta–Merson method to a value η =

η_∞ (η_∞ is a numerical approximation of the mathematical point $\eta = \infty$) at which the asymptotic boundary conditions are met. As the parameters are refined, the quantity η_∞ grows automatically, thus eliminating its effect on the unknown functions. Calculations are considered to be completed if the values of the functions f' and h at "infinity" are about 10^{-10} and f tends to a constant value as $\eta \to \eta_\infty$. It turned out that with these criteria of convergence, η_∞ lie within the range η = 800–1000.

In contrast to the solution of analogous self-similar problems in which a linear dependence of density on temperature is postulated [8-10], a strong dependence of the solution on deficient boundary conditions at $\eta = 0$ (for the considered problems, converging solutions exist within a very narrow range of arbitrarily prescribed values) is a special feature of the numerical integration of (5), (9), (11), and (14).

Results of the calculations are given in Tables 1-4. The same tables present numerical results found earlier [3-7, 10, 11]. Attention should be paid to the substantial difference in the behavior of analogous free-convective flows with linear (n = 1) and square-law (n = 2) dependences of density on temperature. This is true not only for the main velocity and thermal characteristics (n = 2):

a) an isothermal surface:

$$\frac{ux}{\nu} = f'\,\mathrm{Gr}_x^{1/2}, \quad \frac{\tau_w x^2}{\rho\nu^2} = f''(0)\,\mathrm{Gr}_x^{3/4},$$

$$\frac{m}{\mu} = f(\infty)\,\mathrm{Gr}_x^{1/4}, \quad \mathrm{Nu}_x = \{-h'(0)\}\,\mathrm{Gr}_x^{1/4}, \quad \mathrm{Gr}_x = \frac{g\gamma\Delta T_w^2 x^3}{\nu^2};$$

b) an adiabatic plate:

$$\frac{ux}{\nu} = f'\,\mathrm{Gr}_x^{1/3}, \quad \frac{\tau_w x^2}{\rho\nu^2} = f''(0)\,\mathrm{Gr}_x^{1/2},$$

$$\frac{m}{\mu} = f(\infty)\,\mathrm{Gr}_x^{1/6}, \quad \Delta T = \frac{Q_0}{\mu C_p}\,h\,\mathrm{Gr}_x^{-1/6}, \quad \mathrm{Gr}_x = \frac{g\gamma Q_0^2 x^3}{\rho^2 C_p^2 \nu^4};$$

c) a constant heat flux on the surface:

$$\frac{ux}{\nu} = f'\,\mathrm{Gr}_x^{1/3}, \quad \frac{\tau_w x^2}{\rho\nu^2} = f''(0)\,\mathrm{Gr}_x^{1/2},$$

$$\frac{m}{\mu} = f(\infty)\,\mathrm{Gr}_x^{1/6}, \quad \mathrm{Nu}_x = \frac{1}{h(0)}\,\mathrm{Gr}_x^{1/6}, \quad \mathrm{Gr}_x = \frac{g\gamma q_w^2 x^5}{\nu^2 k^2};$$

d) free convection above a linear heat source:

$$\frac{ux}{\nu} = f'\,\mathrm{Gr}_x^{1/3}, \quad \frac{m}{\mu} = f(\infty)\,\mathrm{Gr}_x^{1/6},$$

$$\Delta T = \frac{Q_0}{\mu C_p}\,h\mathrm{Gr}_x^{-1/6}, \quad \mathrm{Gr}_x = \frac{g\gamma Q_0^2 x^3}{\rho^2 C_p^2 \nu^4},$$

TABLE 1. Comparison of Values of $f''(0, \mathrm{Pr})$, $-h'(0, \mathrm{Pr})$, $f'_m(0, \mathrm{Pr})$, $\eta(f'_m)$, and $f(\infty, \mathrm{Pr})$ for the Case of an Isothermal Surface

Pr	[4]	[5]	[6]	[7]	Our data
5	–	–	–	0.53232 0.57296 0.19679 0.982	0.5322756 0.5729631 0.1967988 0.984 0.6438093
6.7	–	–	–	0.49879 0.62417 0.17473 0.946	0.4987514 0.6241446 0.1747261 0.947 0.5935931
7	0.651 0.707	–	–	0.49390 0.63209 0.17159 0.941	0.4938825 0.6320921 0.1716109 0.941 0.5865081
10	–	–	–	0.45569 0.69977 0.14796 0.897	0.4556569 0.6997710 0.1479692 0.898 0.5324935
11.4	0.509 0.615	0.441467 0.723764 0.13896 0.871	0.72610	0.44225 0.72599 0.13999 0.881	0.4422425 0.7260241 0.1400211 0.882 0.5141586
50	–	–	–	0.31272 1.08329 0.07304 0.708	–
100	–	–	–	0.26456 1.29793 0.05299 0.630	0.2649045 1.3002151 0.0533411 0.643 0.2931541

TABLE 2. Comparison of Values of $f''(0, \mathrm{Pr})$, $h(0, \mathrm{Pr})$, $f'_m(0, \mathrm{Pr})$, $\eta(f'_m)$, and $f(\infty, \mathrm{Pr})$ for the Case of an Adiabatic Surface

Pr	[7]	Our data	Pr	[7]	Our data
0.7	0.84292 0.73891 0.64230 1.575 1.99105	–	7	2.18912 1.85529 0.73857 0.757 1.62682	2.1891220 1.8552901 0.7385754 0.759 1.6268063
1	0.95999 0.83863 0.65518 1.426 1.88821	0.9599893 0.8386279 0.6551854 1.427 1.8881967	10	–	2.5873081 2.1794033 0.7542769 0.667 1.6076609
2	–	1.2602547 1.0911458 0.6826771 1.159 1.7507355	11.4	2.75313 2.31408 0.75988 0.634 1.60175	2.7531305 2.3140770 0.7598830 0.635 1.6017350
5	–	1.8759282 1.5993326 0.7233843 0.885 1.6498778	50	–	5.6433241 4.6497670 0.8141457 0.357 1.5609137
6.7	–	2.1451402 1.8194109 0.7366124 0.771 1.6295056	100	7.94751 6.50650 0.83319 0.269 1.55149	7.9475224 6.5065044 0.8331939 0.269 1.5514761

TABLE 3. Values of $f''_m(0, \mathrm{Pr})$, $h(0, \mathrm{Pr})$, $f'_m(\mathrm{Pr})$, $\eta(f'_m)$, and $f(\infty, \mathrm{Pr})$ for the Case of a Constant Heat Flux on the Wall

Pr	$f''(0, \mathrm{Pr})$	$h(0, \mathrm{Pr})$	$f'_m(\mathrm{Pr})$	$\eta(f'_m)$	$f(\infty, \mathrm{Pr})$
5	0.7766409	1.3490768	0.2342305	0.816	0.656030
6.7	0.6684380	1.2748728	0.1964631	0.808	0.587891
7	0.6536396	1.2642436	0.1913436	0.807	0.578433
10	0.5450123	1.1818933	0.1541873	0.796	0.507682
11.4	0.5099208	1.1533939	0.1423717	0.792	0.484226
100	0.1707620	0.7831477	0.0367900	0.701	0.227356

TABLE 4. Comparison of Values of $f'(0, \mathrm{Pr})$, $h(0, \mathrm{Pr})$, and $f(\infty, \mathrm{Pr})$ for the Case of a Linear Heat Source

Pr	[3]	[10]	[11]	Our data
0.01	–	–	0.330113 0.114954 5.251095	–
0.1	–	–	0.507720 0.239243 2.534327	–
1/3	0.58736773 0.33911692 1.87728697	0.5874 0.3391	0.587368 0.339117 1.877309	0.5873677 0.3391169 1.8772870
0.5	–	0.6113 0.3798	0.611333 0.379804 1.749512	–
0.7	–	0.6318 0.4177	0.631743 0.417669 1.673798	–
1	0.65518535 0.46328600 1.61887041	0.6552 0.4633	0.655185 0.463286 1.618874	0.6551853 0.4632860 1.6188704
2	–	0.7096 0.5748	–	–
3	–	0.7478 0.6585	–	–
5	–	0.8022 0.7894	–	0.8052446 0.7861500 1.5792191
6.7	–	–	–	0.8415191 0.8743897 1.5944645
7	–	0.8410 0.8947	0.847186 0.888610 1.597136	0.8471855 0.8886092 1.5971306
10	–	–	–	0.8956044 1.0151754 1.6221215
11.4	–	0.8994 1.0809	0.914406 1.066852 1.632648	0.9144049 1.0668515 1.6326421
100	–	1.1105 2.7727	1.307069 2.518282 1.881156	1.3070689 2.5182830 1.8811542

but also for the effect of the number Pr. For example, fundamentally different results for the velocity field are obtained in the problem of free-convective fluid flow along an adiabatic wall. At $n = 1$, as the Prandtl number increases to 0.2924–0.2934, the maximum longitudinal component of the velocity vector increases, and with a further increase in Pr, a decrease in it is observed [12]. If $n = 2$, the dependence $f'_m(0, \mathrm{Pr})$ has a monotonic character: the maximum of the dimensionless velocity and the friction stress on the wall increase with Pr (Table 2).

Heat transfer on a vertical flat isothermal surface at $n = 2$ (Table 1) is 15% lower than the corresponding value at $n = 1$ [2, 9].

NOTATION

u and v, longitudinal and transverse velocity components; x and y, longitudinal and transverse coordinates; T, temperature; T_w and T_∞, temperatures of the wall and the surroundings; q, heat flux; k, thermal conductivity; ν, kinematic viscosity; μ, dynamic viscosity; $\Delta T = T - T_\infty$, excess temperature; ρ, density; γ, n, parameters of the density; Pr, Prandtl number; C_p, heat capacity at constant pressure; Gr_x, Nu_x, local Grashof and Nusselt numbers; m, mass flow rate per second; g, free-fall acceleration; τ_w, friction stress on the wall. Subscript: m, maximum.

REFERENCES

1. B. Gebhart, Y. Jaluria, R. L. Mahajan, and B. Sammakia, *Buoyancy-Induced Flows and Transport*, Washington (1988).
2. O. G. Martynenko and Yu. A. Sokovishin, *Free-Convective Heat Transfer: Handbook* [in Russian], Minsk (1982).
3. V. N. Korovkin and S. L. Semin, *Vestsi Akad. Navuk BSSR, Ser. Fiz.-Énerg. Navuk*, No. 2, 77-83 (1986).
4. S. L. Goren, *Chem. Eng. Sci.*, **21**, Nos. 6/7, 515-518 (1966).
5. S. Roy, *Indian J. Phys.*, **46**, No. 6, 245-249 (1972).
6. I. Pop and T. Y. Na, *Méc. Appl.*, **33**, No. 4, 303-313 (1988).
7. V. N. Korovkin, in: *Modern Problems of Thermophysics* [in Russian], Novosibirsk (1986), pp. 145-146
8. Yu. A. Sokovishin and O. G. Martynenko, *Introduction to the Theory of Free-Convective Heat Transfer* [in Russian], Leningrad (1982).
9. Y. Jaluria, *Natural Convection* [Russian translation], Moscow (1983).
10. O. G. Martynenko, V. N. Korovkin, and Yu. A. Sokovishin, *Theory of Buoyancy Jets and Wakes* [in Russian], Minsk (1991).
11. V. N. Korovkin and S. N. Prasvet, *Vesti PGU*, **1**, No. 1, 56-63 (1993).
12. V. N. Korovkin and A. P. Andrievskii, *Inzh.-Fiz. Zh.*, **70**, No. 1, 142-145 (1997).